Test 3 Numerical Mathematrics 2
January, 2019

Duration: 1.5 hour.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark for this test.

1. [2] Given the inner product $(f,g) = \int_0^\infty \exp(-x)f(x)g(x)dx$, give the associated first three orthogonal polynomials; so upto degree 2. You do not need to normalize the polynomials.

> **Solution:** The first is simply $\phi_0(x) = 1$. Next we set $\phi_1(x) = x - \alpha$ where α should be chosen such that $(\phi_0, \phi_1) = (1, x-\alpha) = 0$, hence $\alpha = (1,x)/(1,1)$. Here,
>
> $$(x,1) = \int_0^\infty \exp(-x)xdx = -\exp(-x)x|_0^\infty + \int_0^\infty \exp(-x)dx = -\exp(-x)x|_0^\infty = 1$$
>
> and
>
> $$(1,1) = \int_0^\infty \exp(-x)dx = 1$$
>
> So $\alpha = 1$. Next we set $\phi_2(x) = x^2 - \alpha\phi_0(x) - \beta\phi_1(x)$ and require that $(\phi_2, \phi_0) = 0$, $(\phi_2, \phi_1) = 0$. The former gives $\alpha = (x^2, \phi_0)/(\phi_0, \phi_0)$ and the latter $\beta = (x^2, \phi_1)/(\phi_1, \phi_1)$. Now
>
> $$(x^2, \phi_0) = (x^2, 1) = \int_0^\infty \exp(-x)x^2dx = -\exp(-x)x^2|_0^\infty + 2\int_0^\infty x\exp(-x)dx = 2$$
>
> and
>
> $$\begin{aligned}(x^2, \phi_1) &= (x^2, x-1) = (x^2, x) - (x^2, 1) = \int_0^\infty \exp(-x)x^3dx - 2 \\ &= -\exp(-x)x^3|_0^\infty + 3\int_0^\infty x^2\exp(-x)dx - 2 = 3\cdot 2 - 2 = 4\end{aligned}$$
>
> So $\phi_2(x) = x^2 - 2 - 4(x-1) = x^2 - 4x + 2$

2. Suppose that $f(x) = \sum_{i=0}^\infty a_i\phi_i(x)$, where ϕ_i, $i = 0, \cdots$, are orthogonal polynomials in some innerproduct.

   (a) [0.7] Show that $a_i = (f, \phi_i)/(\phi_i, \phi_i)$.

   > **Solution:** Just take the innerproduct of the above with ϕ_k. Then we obtain $(f, \phi_k) = \sum_{i=0}^\infty a_i(\phi_i, \phi_k) = \sum_{i=0}^\infty a_i\delta_{ik}(\phi_k, \phi_k) = a_k(\phi_k, \phi_k)$. Another route follows from minimizing $||f(x) - \sum_{i=0}^\infty a_i\phi_i(x)||$ over the coefficients a_i where the norm is the one associated to the inner product.

   (b) [1] Show that $\sum_{i=0}^\infty a_i^2(\phi_i, \phi_i) = (f, f)$. What is the name of this expression and what does this expression mean?

Solution: We start from the right and substite the expansion given. This yields

$$\begin{aligned}
(f,f) &= (\sum_{i=0}^{\infty} a_i\phi_i(x), \sum_{j=0}^{\infty} a_j\phi_j(x)) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i a_j(\phi_i,\phi_j) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i a_j \delta_{ij}(\phi_i,\phi_i) \\
&= \sum_{i=0}^{\infty} a_i^2(\phi_i,\phi_i)
\end{aligned}$$

This is the Parseval relation. It means that the 2-norm of f is equal to a weighted infinite vector norm over its coefficients.

(c) [0.5] Assume $f(x) = p_n(x)$ where p_n is a polynomial of degree n. Show that $a_i = 0$ for $i > n$.

Solution: Since any polynomial of degree n can be expressed as $\sum_{i=0}^{n} c_i\phi_i(x)$ by just writing out all the polynomials and equating the coefficients of x^k for $k = 0, \cdots, n$) we see that $(f, \phi_i) = 0$ for $i > n$ and hence $a_i = 0$, for $i > n$.

(d) [0.3] What will be the error if we approximate $f(x) = p_n(x)$ by the first $n-1$ terms of the expansion in orthogonal polynomials?

Solution: The error will be $a_n\phi_n(x)$.

(e) [0.5] Suppose we have a finite interval. For which orthogonal polynomials the approximation in the previous part gives us the minimax approximation, i.e., the best polynomial approximation. Explain also why.

Solution: The minimax approximation is found if the error satisfies the equioscillation theorem, i.e., all the $n+1$ extrema should be equally big and alternating in sign. In this case we obtain this if we use Chebyshev polynomials shifted to the considered interval.

(f) [0.5] What do parts (c),(d) and (e) learn us about the best choice to find least squares approximations to the minimax approximation?

Solution: If we have a smooth function in which the coefficients a_i decrease rapidly then using Chebyshev orthogonal polynomials will give us an error which almost satisfies the equioscillation theorem and hence we are close to the minimax solution. For this reason Chebyshev polynomials are the most appropriate choice in general in least squares approximation.

3. [1.5] The error for Gauss-Lobatto interpolation on [-1,1] behaves as $||f - \Pi_{n,w}^{GL} f||_w <$

$Cn^{-s}||f||_{s,w}$ for some $C > 0$ where $w(x) = 1/\sqrt{1-x^2}$. Explain the symbols in this expression and what the expression means.

Solution: $\Pi^{GL}_{n,w} f$ with the weight function $w(x)$ of the Chebyshev polynomials means an interpolation using the zeroes of $T_n(x) + aT_{n-1}(x) + bT_{n-2}(x)$ where a and b are chosen such that the endpoints -1 and 1 of the interval are also zeroes. The norm $||\cdot||_w$ is the norm induced by the innerproduct $(f, g)_w = \int_{-1}^{1} w(x)f(x)g(x)dx$. And $||f||^2_{s,w} \equiv \sum_{i=0}^{s} ||f^{(i)}||^2_w$.
The expression says that when the derivatives of f are bounded in the w-norm up to order s then the convergence of the interpolation behaves like $1/n^s$. So the more derivatives are bounded the faster the convergence will be. This means that for functions for which the norm is arbitrary many times differentiable and at the same time the sum in the definition of $||f||_{s,w}$ is bounded for s to infinity we will see convergence going faster than any expression $1/n^s$, i.e., we find exponential convergence.

4. Consider the ODE $dy/dt = f(t, y(t))$ for $t > t_0$ with $y(t_0) = y_0$.

(a) [0.5] Write this ODE in integral form asuming a constant step size and derive from that the expression one gets if we integrate on a grid from t_n to $t_n + h$, where h is the mesh size.

Solution: Just integrate left and right over time from c to t and we get

$$y(t) - y(c) = \int_c^t f(s, y(s))ds$$

Now we just substitute $c = t_n$ and $t = t_n + h$ and we obtain

$$y(t_n + h) - y(t_n) = \int_{t_n}^{t_n+h} f(t, y(t))dt$$

(b) [1] Suppose we have an integral rule $I(f) \equiv \int_{-1}^{1} f(x)dx \approx I_n(f)$ with an error given by $f^{(m)}(\eta) \int_{-1}^{1} \Pi_{i=0}^{n}(x-x_i)^2 dx$ for an η on [-1,1] and m some positive integer. Show that this error leads to $O(h^{2n+3})$ behavior, if we transform the domain [-1,1] to the domain $[t_n, t_n + h]$.

Solution: Using the transformation $t = t_n+h(x+1)/2$, we obtain that $x-x_i = 2(t - t(x_i))/h$ and consequently

$$\int_{-1}^{1} \Pi_{i=0}^{n}(x - x_i)^2 dx = (2/h)^{2(n+1)+1} \int_{t_n}^{t_n+h} \Pi_{i=0}^{n}(t - t_i)^2 dt$$

where we also used that dx will become $2/hdt$ by the same transformation. So rewriting we have that

$$\int_{t_n}^{t_n+h} \Pi_{i=0}^{n}(t - t_i)^2 dt = (h/2)^{2n+3} \int_{-1}^{1} \Pi_{i=0}^{n}(x - x_i)^2 dx$$

The latter integral will give us just some number independent of h. So indeed we will have an $O(h^{2n+3})$ error in the integrals approximation.

(c) [0.5] Suppose we apply the integral rule to the integral form of the ODE. What is the order of the local truncation error of the resulting difference scheme assuming we can get the needed values of $y(t_i)$ with t_i on $[t_n, t_n + h]$ sufficiently accurate?

Solution: Suppose the integral rule shifted to the interval $[t_n, t_n + h]$ is $\hat{I}_n(f)$. Then, the local truncation error is defined by

$$\begin{aligned}
\tau_n(h) &= \frac{y(t_n + h) - y(t_n) - \hat{I}_n(f)}{h} \\
&= \frac{y(t_n + h) - y(t_n) - \int_{t_n}^{t_n+h} f(t, y(t))dt - O(h^{2n+3})}{h} = O(h^{2n+2})
\end{aligned}$$

where in the last step we have used that y is an exact solution of the ODE and consequently of its integral form.